Comment on "Classifying Novel Phases of Spinor Atoms"

In Bose Einstein condensates of finite spin atoms, both spin-rotation and gauge symmetries can be broken. In a recent paper, Barnett et al [1] classify these finite spin Bose condensates by polyhedra according to the directions of maximally polarized states which are orthogonal to the state ψ under consideration. The purpose of this Comment is to point out that the all important *phase factors* associated with the gauge symmetry has been left out in [1]. In many cases, the Bose condensate is invariant under a rotation only when a suitable accompanying gauge transformation is included. These phase factors also have non-trivial consequences in classification of vortices, thus we obtain results very different from [1].

It suffices to illustrate our point by examples. Consider the ferromagnetic state F of spin 2, with state vector [2] $(1,0,0,0,0)$. Under $R_z(\alpha)$, a rotation about $\hat{z}$ by angle α, the state vector becomes $(e^{-2i\alpha},0,0,0,0)$ and thus *not* invariant (in constrast to [1]). The state-vector is invariant only under the combined operation $R_z(\alpha)e^{2i\alpha}$.

Next, consider the state vector $(i,0,\sqrt{2},0,i)$ belonging to the state named "cyclic" in [2] (named "tetrahedric" in [1], who also chose a different state vector. These state vectors are related by rotation and gauge transformation, as this state is unique [3]) To find the symmetry, it is convenient to note that a state $(\zeta_2,...,\zeta_{-2})$ has the same rotational symmetry as the spatial wavefunction $\psi=\sum_m \zeta_m Y_2^m(\hat{k})$, where Y_l^m are the spherical harmonics. We then find (ignoring overall real proportionality constants irrelevant for discussions here and below) $\psi=\epsilon\hat{k}_x^2+\epsilon^2\hat{k}_y^2+\hat{k}_z^2$ (same form as in [3]), where $\epsilon\equiv e^{2i\pi/3}$. It is easy to see that this state is invariant under two-fold rotations about $\hat{x}$, $\hat{y}$ or $\hat{z}$. Under $2\pi/3$ rotation about diagonals of a cube, e.g., $(\hat{x}+\hat{y}+\hat{z})/\sqrt{3}$, where $(\hat{k}_x,\hat{k}_y,\hat{k}_z)\to(\hat{k}_y,\hat{k}_z,\hat{k}_x)$, the state acquires extra phase factors. Thus the isotropy group of the state is [4, 5] $\{E,3C_2,4C_3\epsilon,4C_3^2\epsilon^2\}$ (named $\mathrm{T(D_2)}$ in [5]) Here E is the identity, and the first $4C_3$ are $2\pi/3$ rotations about $(\pm\hat{x}\pm\hat{y}+\hat{z})/\sqrt{3}$ or $(\mp\hat{x}\pm\hat{y}-\hat{z})/\sqrt{3}$. Note the phase factors ϵ's, which were left out in [1]. (If we also consider time-reversal symmetry Θ, then the isotropy group becomes the larger group $\mathrm{O(D_2)}$: see [5]).

For a third example, consider the state A of spin 3 in [6], with state vector $(1,0,0,0,0,0,1)$. Under $R_z(\alpha)$ the state becomes $(e^{-3i\alpha},0,0,0,0,0,e^{3i\alpha})=e^{-3i\alpha}(1,0,0,0,0,0,e^{6i\alpha})$. Thus under $R_z(\frac{2\pi}{6})$, the *relative* phase between the $m=\pm3$ components is unchanged, but the state acquires an extra factor $e^{-i\pi}$. Hence the invariant operation is $C_6e^{i\pi}$ (not C_6). It follows also that the state is invariant under $C_3=(C_6e^{i\pi})^2$ etc. To find other symmetry operations, we use again the analogy to $l=3$. The wavefunction becomes $-(\hat{k}_x+i\hat{k}_y)^3+(\hat{k}_x-i\hat{k}_y)^3$. It is evident that the state is invariant under π rotation about $\hat{y}$, and also under π rotation about $\hat{x}$ except a phase factor $e^{i\pi}$. The existence of C_3 tells us that there are two other horizontal two-fold axis $2\pi/3$ with respect to each of these. The isotropy group for this state is thus $\{E,2C_3,2C_6e^{i\pi},C_2e^{i\pi},3U_2e^{i\pi},3U_2'\}$. (named $\mathrm{D_6(D_3)}$ in [5]). Note the phase factors $e^{i\pi}$ accompanying C_6, C_2 and U_2's, whereas [1] simply represented this state as a hexagon.

These phase factors in the isotropy groups have non-trivial consequences when classifying vortices. As an example, for the cyclic state, vortices are divided into [4, 7] *seven* classes in additional to the circulation numbers n. (In [1] however, it was stated that the number of topological excitations is six, and circulation numbers were left out.) We note further that, for the vortices where the order parameter is rotated by C_3 $[C_3^2]$ when one travels along a path encircling that vortex, the associated phase changes should be (see the elements in $\mathrm{T(D_2)}$) $(2n+\frac{2}{3})\pi$ $[\,(2n+\frac{4}{3})\pi]$, not the ordinary $2n\pi$. These phase factors must be kept correctly to properly discuss combination of two vortices [4]. For example, combining two vortices with circulations $(2n_1+\frac{2}{3})\pi$ and $(2n_2+\frac{4}{3})\pi$ leads to a total circulation of $2(n_1+n_2+1)\pi$ but not $2(n_1+n_2)\pi$.

In conclusion, we have pointed out that [1] has left out phase factors in their discussions on symmetries and vortices of spin condensates. More discussions on symmetries of these states can be found in [7].

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